

Injective Envelopes and (Gorenstein) Flat Covers^{*†}

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Abstract

In terms of the duality property of injective preenvelopes and flat precovers, we get an equivalent characterization of left Noetherian rings. For a left and right Noetherian ring R , we prove that the flat dimension of the injective envelope of any (Gorenstein) flat left R -module is at most the flat dimension of the injective envelope of ${}_R R$. Then we get that the injective envelope of ${}_R R$ is (Gorenstein) flat if and only if the injective envelope of every Gorenstein flat left R -module is (Gorenstein) flat, if and only if the injective envelope of every flat left R -module is (Gorenstein) flat, if and only if the (Gorenstein) flat cover of every injective left R -module is injective, and if and only if the opposite version of one of these conditions is satisfied.

1. Introduction

Throughout this paper, all rings are associative with identity. For a ring R , we use $\text{Mod } R$ (resp. $\text{Mod } R^{op}$) to denote the category of left (resp. right) R -modules.

It is well known that (pre)envelopes and (pre)covers of modules are dual notions. These are fundamental and important in relative homological algebra. Also note that coherent (resp. Noetherian) rings can be characterized by the equivalence of the absolutely purity (resp. injectivity) of modules and the flatness of their character modules (see [ChS]). In this paper, under some conditions, we show that we get a precover after applying a contravariant Hom functor associated with a bimodule to a preenvelope. Then we obtain an equivalent characterization of coherent (resp. Noetherian) rings in terms of the duality property between absolutely pure (resp. injective) preenvelopes and flat precovers.

Bican, El Bashir and Enochs proved in [BEE] that every module has a flat cover for any ring. Furthermore, Enochs, Jenda and Lopez-Ramos proved in [EJL] that every left module has a Gorenstein flat cover for a right coherent ring. On the other hand, we know

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from [B] that the injective envelope of R is flat for a commutative Gorenstein ring R . In [ChE] Cheatham and Enochs proved that for a commutative Noetherian ring R , the injective envelope of R is flat if and only if the injective envelope of every flat R -module is flat. In this paper, over a left and right Noetherian ring R , we will characterize when the injective envelope of ${}_R R$ is (Gorenstein) flat in terms of the (Gorenstein) flatness of the injective envelopes of (Gorenstein) flat left R -modules and the injectivity of the (Gorenstein) flat covers of injective left R -modules.

This paper is organized as follows.

In Section 2, we give some terminology and some preliminary results. In particular, as generalizations of strongly cotorsion modules and strongly torsionfree modules, we introduce the notions of n -(Gorenstein) cotorsion modules and n -(Gorenstein) torsionfree modules, and then establish a duality relation between right n -(Gorenstein) torsionfree modules and left n -(Gorenstein) cotorsion modules.

Let R and S be rings and let ${}_S U_R$ be a given left S -right R -bimodule. For a subcategory \mathcal{X} of $\text{Mod } S$ (or $\text{Mod } R^{op}$), we denote by $\mathcal{X}^* = \{X^* \mid X \in \mathcal{X}\}$, where $(-)^* = \text{Hom}(-, {}_S U_R)$. In Section 3, we prove that if \mathcal{C} is a subcategory of $\text{Mod } S$ and \mathcal{D} is a subcategory of $\text{Mod } R^{op}$ such that $\mathcal{C}^* \subseteq \mathcal{D}$ and $\mathcal{D}^* \subseteq \mathcal{C}$, then a homomorphism $f : A \rightarrow C$ in $\text{Mod } S$ being a \mathcal{C} -preenvelope of A implies that $f^* : C^* \rightarrow A^*$ is a \mathcal{D} -precover of A^* in $\text{Mod } R^{op}$. As applications of this result, we get that a ring R is left coherent if and only if a monomorphism $f : A \rightarrow E$ in $\text{Mod } R$ being an absolutely pure preenvelope of A implies that $f^+ : E^+ \rightarrow A^+$ is a flat precover of A^+ in $\text{Mod } R^{op}$, and that R is left Noetherian if and only if a monomorphism $f : A \rightarrow E$ in $\text{Mod } R$ being an injective preenvelope of A is equivalent to $f^+ : E^+ \rightarrow A^+$ being a flat precover of A^+ in $\text{Mod } R^{op}$, where $(-)^+$ is the character functor.

By using the result about the equivalent characterization of Noetherian rings obtained in Section 3, we first prove in Section 4 that the flat dimension of the injective envelope of any (Gorenstein) flat left R -module is at most the flat dimension of the injective envelope of ${}_R R$ for a left and right Noetherian ring R . Then we investigate the relation between the injective envelopes of (Gorenstein) flat modules and (Gorenstein) flat covers of injective modules. We give a list of equivalent conditions relating these notions. For a left and right Noetherian ring R , we prove that the injective envelope of ${}_R R$ is (Gorenstein) flat if and only if the injective envelope of every Gorenstein flat left R -module is (Gorenstein) flat, if and only if the injective envelope of every flat left R -module is (Gorenstein) flat, if and only if the (Gorenstein) flat cover of every injective left R -module is injective, and if and only if the opposite version of one of these conditions is satisfied.

2. Preliminaries

In this section, we give some terminology and some preliminary results for later use.

Definition 2.1 ([E]) Let R be a ring and \mathcal{C} a subcategory of $\text{Mod } R$. The homomorphism $f : C \rightarrow D$ in $\text{Mod } R$ with $C \in \mathcal{C}$ is said to be a \mathcal{C} -precover of D if for any homomorphism $g : C' \rightarrow D$ in $\text{Mod } R$ with $C' \in \mathcal{C}$, there exists a homomorphism $h : C' \rightarrow C$ such that the following diagram commutes:

$$\begin{array}{ccc} & C' & \\ \swarrow h & \downarrow g & \\ C & \xrightarrow{f} & D \end{array}$$

The homomorphism $f : C \rightarrow D$ is said to be *right minimal* if an endomorphism $h : C \rightarrow C$ is an automorphism whenever $f = fh$. A \mathcal{C} -precover $f : C \rightarrow D$ is called a \mathcal{C} -cover if f is right minimal. Dually, the notions of a \mathcal{C} -preenvelope, a *left minimal homomorphism* and a \mathcal{C} -envelope are defined.

We begin with the following easy observation.

Lemma 2.2 Let R be a ring and \mathcal{D} a subcategory of $\text{Mod } R^{op}$, which is closed under direct products. If $f_i : D_i \rightarrow M_i$ is a \mathcal{D} -precover of M_i in $\text{Mod } R^{op}$ for any $i \in I$, where I is an index set, then $\prod_{i \in I} f_i : \prod_{i \in I} D_i \rightarrow \prod_{i \in I} M_i$ is a \mathcal{D} -precover of $\prod_{i \in I} M_i$.

Proof. Assume that $f_i : D_i \rightarrow M_i$ is a \mathcal{D} -precover of M_i in $\text{Mod } R^{op}$ for any $i \in I$. Because \mathcal{D} is closed under direct products by assumption, we get a homomorphism $\prod_{i \in I} f_i : \prod_{i \in I} D_i \rightarrow \prod_{i \in I} M_i$ in $\text{Mod } R^{op}$ with $\prod_{i \in I} D_i \in \mathcal{D}$.

Let $g : H \rightarrow \prod_{i \in I} M_i$ be a homomorphism in $\text{Mod } R^{op}$ with $H \in \mathcal{D}$. Then there exists a homomorphism $h_i : H \rightarrow M_i$ such that $f_i h_i = \pi_i g$, where π_i is the i th projection of $\prod_{i \in I} M_i$ for any $i \in I$. Put $h = \prod_{i \in I} h_i$. Then we have that $(\prod_{i \in I} f_i)h = (\prod_{i \in I} f_i)(\prod_{i \in I} h_i) = \prod_{i \in I} (f_i h_i) = \prod_{i \in I} (\pi_i g) = g$, that is, we get a homomorphism $h : H \rightarrow \prod_{i \in I} D_i$ such that the following diagram commutes:

$$\begin{array}{ccc} & H & \\ \swarrow h & \downarrow g & \\ \prod_{i \in I} D_i & \xrightarrow{\prod_{i \in I} f_i} & \prod_{i \in I} M_i \end{array}$$

Thus $\prod_{i \in I} f_i : \prod_{i \in I} D_i \rightarrow \prod_{i \in I} M_i$ is a \mathcal{D} -precover of $\prod_{i \in I} M_i$. □

The following useful result is usually called Wakamatsu's lemma.

Lemma 2.3 ([X, Lemma 2.1.1]) Let R be a ring and \mathcal{C} a subcategory of $\text{Mod } R$, which

is closed under extensions. If $f : C \rightarrow D$ is a \mathcal{C} -cover of a module D in $\text{Mod } R$, then $\text{Ext}_R^1(C', \text{Ker } f) = 0$ for any $C' \in \mathcal{C}$.

As generalizations of flat modules and the flat dimension of modules, the notions of Gorenstein flat modules and the Gorenstein flat dimension of modules were introduced by Enochs, Jenda and Torrecillas in [EJT] and by Holm in [H], respectively.

Definition 2.4 ([EJT] and [H]) Let R be a ring. A module M in $\text{Mod } R$ is called *Gorenstein flat* if there exists an exact sequence:

$$\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$$

in $\text{Mod } R$ with all terms flat, such that $M = \text{Im}(F_0 \rightarrow F^0)$ and the sequence is still exact after applying the functor $I \otimes_R -$ for any injective right R -module I . The *Gorenstein flat dimension* of M is defined as $\inf\{n \mid \text{there exists an exact sequence } 0 \rightarrow G_n \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0 \text{ in } \text{Mod } R \text{ with } G_i \text{ Gorenstein flat for any } 0 \leq i \leq n\}$.

Lemma 2.5 For a ring R , an injective left (or right) R -module is flat if and only if it is Gorenstein flat.

Proof. The necessity is trivial. Notice that any Gorenstein flat module can be embedded into a flat module, so it is easy to see that a Gorenstein flat module is flat if it is injective. \square

For a ring R , we use $\mathcal{GF}(R)$ to denote the subcategory of $\text{Mod } R$ consisting of Gorenstein flat modules.

Lemma 2.6 ([H, Theorem 3.7]) For a right coherent ring R , $\mathcal{GF}(R)$ is closed under extensions and under direct summands.

Definition 2.7 ([X]) Let R be a ring. A module M in $\text{Mod } R$ is called *strongly cotorsion* if $\text{Ext}_R^1(X, M) = 0$ for any $X \in \text{Mod } R$ with finite flat dimension. A module N in $\text{Mod } R^{op}$ is called *strongly torsionfree* if $\text{Tor}_1^R(N, X) = 0$ for any $X \in \text{Mod } R$ with finite flat dimension.

We generalize these notions and introduce the notions of n -cotorsion modules and n -torsionfree modules and that of n -Gorenstein cotorsion modules and n -Gorenstein torsionfree modules as follows.

Definition 2.8 Let R be a ring and n a positive integer.

(1) A module M in $\text{Mod } R$ is called *n -cotorsion* if $\text{Ext}_R^1(X, M) = 0$ for any $X \in \text{Mod } R$ with flat dimension at most n . A module N in $\text{Mod } R^{op}$ is called *n -torsionfree* if $\text{Tor}_1^R(N, X) = 0$ for any $X \in \text{Mod } R$ with flat dimension at most n .

(2) A module M in $\text{Mod } R$ is called *n-Gorenstein cotorsion* if $\text{Ext}_R^1(X, M) = 0$ for any $X \in \text{Mod } R$ with Gorenstein flat dimension at most n ; and M is called *strongly Gorenstein cotorsion* if it is *n-Gorenstein cotorsion* for all n . A module N in $\text{Mod } R^{op}$ is called *n-Gorenstein torsionfree* if $\text{Tor}_1^R(N, X) = 0$ for any $X \in \text{Mod } R$ with Gorenstein flat dimension at most n ; and N is called *strongly Gorenstein torsionfree* if it is *n-Gorenstein torsionfree* for all n .

Remark. (1) We have the descending chains: $\{1\text{-cotorsion modules}\} \supseteq \{2\text{-cotorsion modules}\} \supseteq \cdots \supseteq \{\text{strongly cotorsion modules}\}$, and $\{1\text{-torsionfree modules}\} \supseteq \{2\text{-torsionfree modules}\} \supseteq \cdots \supseteq \{\text{strongly torsionfree modules}\}$. In particular, $\{\text{strongly cotorsion modules}\} = \bigcap_{n \geq 1} \{n\text{-cotorsion modules}\}$ and $\{\text{strongly torsionfree modules}\} = \bigcap_{n \geq 1} \{n\text{-torsionfree modules}\}$.

(2) Similarly, we have the descending chains: $\{1\text{-Gorenstein cotorsion modules}\} \supseteq \{2\text{-Gorenstein cotorsion modules}\} \supseteq \cdots \supseteq \{\text{strongly Gorenstein cotorsion modules}\}$, and $\{1\text{-Gorenstein torsionfree modules}\} \supseteq \{2\text{-Gorenstein torsionfree modules}\} \supseteq \cdots \supseteq \{\text{strongly Gorenstein torsionfree modules}\}$. In particular, $\{\text{strongly Gorenstein cotorsion modules}\} = \bigcap_{n \geq 1} \{n\text{-Gorenstein cotorsion modules}\}$ and $\{\text{strongly Gorenstein torsionfree modules}\} = \bigcap_{n \geq 1} \{n\text{-Gorenstein torsionfree modules}\}$.

Lemma 2.9 ([CE, p.120, Proposition 5.1]) *Let R and S be rings with the condition $({}_R A, {}_S B_R, {}_S C)$. If ${}_S C$ is injective, then we have the following isomorphism of Abelian groups for any $n \geq 1$:*

$$\text{Hom}_S(\text{Tor}_n^R(B, A), C) \cong \text{Ext}_R^n(A, \text{Hom}_S(B, C)).$$

We denote by $(-)^+ = \text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where \mathbb{Z} is the additive group of integers and \mathbb{Q} is the additive group of rational numbers. The following result establishes a duality relation between right *n*-(Gorenstein) torsionfree modules and left *n*-(Gorenstein) cotorsion modules.

Proposition 2.10 *Let R be a ring and N a module in $\text{Mod } R^{op}$. Then for any $n \geq 1$, we have*

(1) *N is n -torsionfree if and only if N^+ is n -cotorsion. In particular, N is strongly torsionfree if and only if N^+ is strongly cotorsion.*

(2) *N is n -Gorenstein torsionfree if and only if N^+ is n -Gorenstein cotorsion. In particular, N is strongly Gorenstein torsionfree if and only if N^+ is strongly Gorenstein cotorsion.*

Proof. By Lemma 2.9, we have that

$$[\mathrm{Tor}_1^R(N, A)]^+ \cong \mathrm{Ext}_R^1(A, N^+)$$

for any $A \in \mathrm{Mod} R$ and $N \in \mathrm{Mod} R^{op}$. Then both assertions follow easily. \square

The following two lemmas are well known.

Lemma 2.11 ([F, Theorem 2.1]) *For a ring R , the flat dimension of M and the injective dimension of M^+ are identical for any $M \in \mathrm{Mod} R$ (resp. $\mathrm{Mod} R^{op}$).*

For a ring R , recall from [M] that a module M in $\mathrm{Mod} R$ is called *absolutely pure* if it is a pure submodule in every module in $\mathrm{Mod} R$ that contains it, or equivalently, if it is pure in every injective module in $\mathrm{Mod} R$ that contains it. Absolutely pure modules are also known as *FP-injective modules*. It is trivial that an injective module is absolutely pure. By [M, Theorem 3], a ring R is left Noetherian if and only if every absolutely pure module in $\mathrm{Mod} R$ is injective.

Lemma 2.12 (1) ([ChS, Theorem 1]) *A ring R is left coherent if and only if a module A in $\mathrm{Mod} R$ being absolutely pure is equivalent to A^+ being flat in $\mathrm{Mod} R^{op}$.*

(2) ([ChS, Theorem 2] and [F, Theorem 2.2]) *A ring R is left (resp. right) Noetherian if and only if a module E in $\mathrm{Mod} R$ (resp. $\mathrm{Mod} R^{op}$) being injective is equivalent to E^+ being flat in $\mathrm{Mod} R^{op}$ (resp. $\mathrm{Mod} R$), and if and only if the injective dimension of M and the flat dimension of M^+ are identical for any $M \in \mathrm{Mod} R$ (resp. $\mathrm{Mod} R^{op}$).*

As a generalization of injective modules, the notion of Gorenstein injective modules was introduced by Enochs and Jenda in [EJ1] as follows.

Definition 2.13 ([EJ1]) Let R be a ring. A module M in $\mathrm{Mod} R$ is called *Gorenstein injective* if there exists an exact sequence:

$$\cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$$

in $\mathrm{Mod} R$ with all terms injective, such that $M = \mathrm{Im}(I_0 \rightarrow I^0)$ and the sequence is still exact after applying the functor $\mathrm{Hom}_R(I, -)$ for any injective left R -module I .

The following result is a Gorenstein version of Lemmas 2.11 and 2.12(2).

Lemma 2.14 (1) ([H, Theorem 3.6]) *For a ring R , if F is Gorenstein flat in $\mathrm{Mod} R$ (resp. $\mathrm{Mod} R^{op}$), then F^+ is Gorenstein injective in $\mathrm{Mod} R^{op}$ (resp. $\mathrm{Mod} R$).*

(2) ([EJ2, Corollary 10.3.9]) *Let R be a Gorenstein ring (that is, R is a left and right Noetherian ring with finite left and right self-injective dimensions). If Q is Gorenstein injective in $\mathrm{Mod} R$ (resp. $\mathrm{Mod} R^{op}$), then Q^+ is Gorenstein flat in $\mathrm{Mod} R^{op}$ (resp. $\mathrm{Mod} R$).*

3. The duality between preenvelopes and precovers

In this section, we study the duality properties between preenvelopes and precovers.

Let R and S be rings and let ${}_S U_R$ be a given left S -right R -bimodule. We denote by $(-)^* = \text{Hom}(-, {}_S U_R)$. For a subcategory \mathcal{X} of $\text{Mod } S$ (or $\text{Mod } R^{op}$), we denote by $\mathcal{X}^* = \{X^* \mid X \in \mathcal{X}\}$. For any $X \in \text{Mod } S$ (or $\text{Mod } R^{op}$), $\sigma_X : X \rightarrow X^{**}$ defined by $\sigma_X(x)(f) = f(x)$ for any $x \in X$ and $f \in X^*$ is the canonical evaluation homomorphism.

Theorem 3.1 *Let \mathcal{C} be a subcategory of $\text{Mod } S$ and \mathcal{D} a subcategory of $\text{Mod } R^{op}$ such that $\mathcal{C}^* \subseteq \mathcal{D}$ and $\mathcal{D}^* \subseteq \mathcal{C}$. If $f : A \rightarrow C$ is a \mathcal{C} -preenvelope of a module A in $\text{Mod } S$, then $f^* : C^* \rightarrow A^*$ is a \mathcal{D} -precover of A^* in $\text{Mod } R^{op}$.*

Proof. Assume that $f : A \rightarrow C$ is a \mathcal{C} -preenvelope of a module A in $\text{Mod } S$. Then we have a homomorphism $f^* : C^* \rightarrow A^*$ in $\text{Mod } R^{op}$ with $C^* \in \mathcal{C}^* \subseteq \mathcal{D}$. Let $g : D \rightarrow A^*$ be a homomorphism in $\text{Mod } R^{op}$ with $D \in \mathcal{D}$. Then $D^* \in \mathcal{D}^* \subseteq \mathcal{C}$.

Consider the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \sigma_A \downarrow & & \\ A^{**} & & \\ g^* \downarrow & & \\ D^* & & \end{array}$$

Because $f : A \rightarrow C$ is a \mathcal{C} -preenvelope of A , there exists a homomorphism $h : C \rightarrow D^*$ such that the above diagram commutes, that is, $hf = g^* \sigma_A$. Then we have $\sigma_A^* g^{**} = f^* h^*$. On the other hand, we have the following commutative diagram:

$$\begin{array}{ccc} D & \xrightarrow{g} & A^* \\ \sigma_D \downarrow & & \downarrow \sigma_{A^*} \\ D^{**} & \xrightarrow{g^{**}} & A^{***} \end{array}$$

that is, we have $\sigma_{A^*} g = g^{**} \sigma_D$. By [AF, Proposition 20.14], $\sigma_A^* \sigma_{A^*} = 1_{A^*}$. So we have that $g = 1_{A^*} g = \sigma_A^* \sigma_{A^*} g = \sigma_A^* g^{**} \sigma_D = f^* (h^* \sigma_D)$, that is, we get a homomorphism $h^* \sigma_D : D \rightarrow C^*$ such that the following diagram commutes:

$$\begin{array}{ccc} & D & \\ h^* \sigma_D \swarrow & \downarrow g & \searrow \\ C^* & \xrightarrow{f^*} & A^* \end{array}$$

Thus $f^* : C^* \rightarrow A^*$ is a \mathcal{D} -precover of A^* . □

In the rest of this section, we will give some applications of Theorem 3.1.

For a ring R and a subcategory \mathcal{X} of $\text{Mod } R$ (or $\text{Mod } R^{op}$), we denote by $\mathcal{X}^+ = \{X^+ \mid X \in \mathcal{X}\}$.

Corollary 3.2 (1) *Let \mathcal{C} be a subcategory of $\text{Mod } R$ and \mathcal{D} a subcategory of $\text{Mod } R^{op}$ such that $\mathcal{C}^+ \subseteq \mathcal{D}$ and $\mathcal{D}^+ \subseteq \mathcal{C}$. If $f : A \rightarrow C$ is a \mathcal{C} -preenvelope of a module A in $\text{Mod } R$, then $f^+ : C^+ \rightarrow A^+$ is a \mathcal{D} -precover of A^+ in $\text{Mod } R^{op}$.*

(2) *Let $f : M \rightarrow N$ be a homomorphism in $\text{Mod } R$. If $f^+ : N^+ \rightarrow M^+$ is left (resp. right) minimal, then f is right (resp. left) minimal.*

Proof. (1) Notice that $\text{Hom}_R(-, R^+) \cong (-)^+$ by the adjoint isomorphism theorem, so the assertion is an immediate consequence of Theorem 3.1.

(2) Assume that $f^+ : N^+ \rightarrow M^+$ is left minimal. If $f : M \rightarrow N$ is not right minimal, then there exists an endomorphism $h : M \rightarrow M$ such that $f = fh$ but h is not an automorphism. So $f^+ = h^+ f^+$ and h^+ is not an automorphism. It follows that f^+ is not left minimal, which is a contradiction. Thus we conclude that f is right minimal. Similarly, we get that f is left minimal if f^+ is right minimal. \square

We use $\mathcal{AP}(R)$, $\mathcal{F}(R)$ and $\mathcal{I}(R)$ to denote the subcategories of $\text{Mod } R$ consisting of absolutely pure modules, flat modules and injective modules, respectively. Recall that an $\mathcal{F}(R)$ -preenvelope is called a *flat preenvelope*. By [E, Proposition 5.1] or [EJ2, Proposition 6.5.1], we have that R is a left coherent ring if and only if every module in $\text{Mod } R^{op}$ has a flat preenvelope. Also recall that an $\mathcal{AP}(R)$ -precover and an $\mathcal{I}(R)$ -precover are called an *absolutely pure precover* and an *injective precover*, respectively. It is known that every module in $\text{Mod } R$ has an absolutely pure precover for a left coherent ring R by [P, Theorem 2.6], and every module in $\text{Mod } R$ has an injective precover for a left Noetherian ring R by [E, Proposition 2.2].

Corollary 3.3 *Let R be a left coherent ring. If a homomorphism $f : A \rightarrow F$ in $\text{Mod } R^{op}$ is a flat preenvelope of A , then $f^+ : F^+ \rightarrow A^+$ is an absolutely pure precover and an injective precover of A^+ in $\text{Mod } R$.*

Proof. By Lemmas 2.11 and 2.12(1) and the opposite version of Corollary 3.2(1). \square

We use $\mathcal{GI}(R)$ to denote the subcategory of $\text{Mod } R$ consisting of Gorenstein injective modules. Recall that a $\mathcal{GI}(R)$ -precover (resp. preenvelope) and a $\mathcal{GF}(R)$ -(pre)cover (resp. preenvelope) are called a *Gorenstein injective precover* (resp. *preenvelope*) and a *Gorenstein flat (pre)cover* (resp. *preenvelope*), respectively. Let R be a Gorenstein ring. Then every

module in $\text{Mod } R$ has a Gorenstein injective preenvelope and a Gorenstein flat preenvelope by [EJ2, Theorems 11.2.1 and 11.8.2], and every module in $\text{Mod } R^{op}$ has a Gorenstein injective precover and a Gorenstein flat precover by [EJ2, Theorems 11.1.1] and [EJL, Theorem 2.11], respectively.

Corollary 3.4 *Let R be a Gorenstein ring.*

(1) *If a monomorphism $f : A \rightarrowtail Q$ in $\text{Mod } R$ is a Gorenstein injective preenvelope of A , then $f^+ : Q^+ \rightarrow A^+$ is a Gorenstein flat precover of A^+ in $\text{Mod } R^{op}$.*

(2) *If a homomorphism $f : A \rightarrow G$ in $\text{Mod } R$ is a Gorenstein flat preenvelope of A , then $f^+ : G^+ \rightarrow A^+$ is a Gorenstein injective precover of A^+ in $\text{Mod } R^{op}$.*

Proof. By Lemma 2.14 and Corollary 3.2(1). □

Recall that an $\mathcal{AP}(R)$ -preenvelope is called an *absolutely pure preenvelope*. By [EJ2, Proposition 6.2.4], every module in $\text{Mod } R$ has an absolutely pure preenvelope. Recall that an $\mathcal{I}(R)$ -(pre)envelope and an $\mathcal{F}(R)$ -(pre)cover are called an *injective (pre)envelope* and a *flat (pre)cover*, respectively. By [BEE, Theorem 3], every module in $\text{Mod } R$ has a flat cover. In the following result, we give an equivalent characterization of left coherent rings in terms of the duality property between absolutely pure preenvelopes and flat precovers.

Theorem 3.5 *Consider the following conditions for a ring R .*

(1) *R is a left coherent ring.*

(2) *If a monomorphism $f : A \rightarrowtail E$ in $\text{Mod } R$ is an absolutely pure preenvelope of A , then $f^+ : E^+ \rightarrow A^+$ is a flat precover of A^+ in $\text{Mod } R^{op}$.*

(3) *If a monomorphism $f : A \rightarrowtail E$ in $\text{Mod } R$ is an injective preenvelope of A , then $f^+ : E^+ \rightarrow A^+$ is a flat precover of A^+ in $\text{Mod } R^{op}$.*

We have $(1) \Leftrightarrow (2) \Rightarrow (3)$.

Proof. (1) \Rightarrow (2) and (1) \Rightarrow (3) follow from Lemmas 2.11 and 2.12(1) and Corollary 3.2(1).

(2) \Rightarrow (1) Note that a left R -module E is absolutely pure if E^+ is flat (see [ChS]). So by (2), we have that a left R -module E is absolutely pure if and only if E^+ is flat. Then it follows from Lemma 2.12(1) that R is a left coherent ring. □

The following example illustrates that for a left (and right) coherent ring R , neither of the converses of (2) and (3) in Theorem 3.5 hold true in general.

Example 3.6 Let R be a von Neumann regular ring but not a semisimple Artinian ring. Then R is a left and right coherent ring and every left R -module is absolutely pure by [M,

Theorem 5]. So both (2) and (3) in Theorem 3.5 hold true. Because R is not a semisimple Artinian ring, there exists a non-injective left R -module M . Then we have

(1) The injective envelope $\alpha : M \hookrightarrow E^0(M)$ is not an absolutely pure preenvelope, because there does not exist a homomorphism $E^0(M) \rightarrow M$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & E^0(M) \\ 1_M \downarrow & & \\ M & & \end{array}$$

So the converse of (2) in Theorem 3.5 does not hold true.

(2) Note that the identity homomorphism $1_M : M \rightarrow M$ is an absolutely pure envelope of M but not an injective preenvelope of M . So the converse of (3) in Theorem 3.5 does not hold true.

Thus we also get that neither of the converses of Corollary 3.2(1) and Theorem 3.1 hold true in general.

In the following result, we give an equivalent characterization of left Noetherian rings in terms of the duality property between injective preenvelopes and flat precovers.

Theorem 3.7 *The following statements are equivalent for a ring R .*

(1) *R is a left Noetherian ring.*

(2) *A monomorphism $f : A \hookrightarrow E$ in $\text{Mod } R$ is an injective preenvelope of A if and only if $f^+ : E^+ \twoheadrightarrow A^+$ is a flat precover of A^+ in $\text{Mod } R^{op}$.*

If one of the above equivalent conditions is satisfied, then $f^+ : E^+ \twoheadrightarrow A^+$ being a flat cover of A^+ implies that $f : A \hookrightarrow E$ is an injective envelope of A .

Proof. (2) \Rightarrow (1) follows from Lemma 2.12(2).

(1) \Rightarrow (2) Assume that R is a left Noetherian ring. If $f : A \hookrightarrow E$ in $\text{Mod } R$ is an injective preenvelope of A , then $f^+ : E^+ \twoheadrightarrow A^+$ is a flat precover of A^+ by Theorem 3.5. Conversely, if $f^+ : E^+ \twoheadrightarrow A^+$ is a flat precover of A^+ in $\text{Mod } R^{op}$, then E is injective by Lemma 2.12(2), and so $f : A \hookrightarrow E$ is an injective preenvelope of A .

The last assertion follows from (2) and Corollary 3.2(2). \square

4. Injective envelopes of (Gorenstein) flat modules

In this section, R is a left and right Noetherian ring. We will investigate the relation between the injective envelopes of (Gorenstein) flat modules and (Gorenstein) flat covers of injective modules.

For a module M in $\text{Mod } R$, we denote the injective envelope and the flat cover of M by $E^0(M)$ and $F_0(M)$ respectively, and denote the injective dimension and the flat dimension of M by $\text{id}_R M$ and $\text{fd}_R M$ respectively.

Theorem 4.1 $\text{fd}_R E^0({}_R R) = \sup\{\text{fd}_R E^0(F) \mid F \in \text{Mod } R \text{ is flat}\} = \sup\{\text{fd}_R E^0(G) \mid G \in \text{Mod } R \text{ is Gorenstein flat}\}.$

Proof. It is trivial that $\text{fd}_R E^0({}_R R) \leq \sup\{\text{fd}_R E^0(F) \mid F \in \text{Mod } R \text{ is flat}\} \leq \sup\{\text{fd}_R E^0(G) \mid G \in \text{Mod } R \text{ is Gorenstein flat}\}.$ So it suffices to prove the opposite inequalities.

We first prove $\text{fd}_R E^0({}_R R) \geq \sup\{\text{fd}_R E^0(F) \mid F \in \text{Mod } R \text{ is flat}\}.$ Without loss of generality, suppose $\text{fd}_R E^0({}_R R) = n < \infty.$ Because R is a left Noetherian ring and ${}_R R \hookrightarrow E^0({}_R R)$ is an injective envelope of ${}_R R$, $[E^0({}_R R)]^+ \rightarrow ({}_R R)^+$ is a flat precover of $({}_R R)^+$ in $\text{Mod } R^{op}$ by Theorem 3.7.

Let $F \in \text{Mod } R$ be flat. Then F^+ is injective in $\text{Mod } R^{op}$ by Lemma 2.11. Because $({}_R R)^+$ is an injective cogenerator for $\text{Mod } R^{op}$ by [S, p.32, Proposition 9.3], F^+ is isomorphic to a direct summand of $\prod_{i \in I} ({}_R R)^+$ for some set $I.$ So $F_0(F^+)$ is isomorphic to a direct summand of any flat precover of $\prod_{i \in I} ({}_R R)^+.$ Notice that $\mathcal{F}(R^{op})$ is closed under direct products by [C, Theorem 2.1], so $\prod_{i \in I} [E^0({}_R R)]^+ \rightarrow \prod_{i \in I} ({}_R R)^+$ is a flat precover of $\prod_{i \in I} ({}_R R)^+$ by the above argument and Lemma 2.2. Thus we get that $F_0(F^+)$ is isomorphic to a direct summand of $\prod_{i \in I} [E^0({}_R R)]^+.$ Because $\text{fd}_R E^0({}_R R) = n,$ $\text{id}_{R^{op}} [E^0({}_R R)]^+ = n$ by Lemma 2.11. So $\text{id}_{R^{op}} \prod_{i \in I} [E^0({}_R R)]^+ = n$ and $\text{id}_{R^{op}} F_0(F^+) \leq n.$

Because R is a right Noetherian ring, we get an injective preenvelope $F^{++} \hookrightarrow [F_0(F^+)]^+$ of F^{++} with $\text{fd}_R [F_0(F^+)]^+ \leq n$ by Lemma 2.12(2). By [S, p.48, Exercise 41], there exists a monomorphism $F \hookrightarrow F^{++}.$ So $E^0(F)$ is isomorphic to a direct summand of $E^0(F^{++}),$ and hence a direct summand of $[F_0(F^+)]^+.$ It follows that $\text{fd}_R E^0(F) \leq n.$ Thus we get that $\text{fd}_R E^0({}_R R) \geq \sup\{\text{fd}_R E^0(F) \mid F \in \text{Mod } R \text{ is flat}\}.$

Next, we prove $\sup\{\text{fd}_R E^0(F) \mid F \in \text{Mod } R \text{ is flat}\} \geq \sup\{\text{fd}_R E^0(G) \mid G \in \text{Mod } R \text{ is Gorenstein flat}\}.$ Let $G \in \text{Mod } R$ be Gorenstein flat. Then G can be embedded into a flat left R -module $F,$ which implies that $E^0(G)$ is isomorphic to a direct summand of $E^0(F)$ and $\text{fd}_R E^0(G) \leq \text{fd}_R E^0(F).$ Thus the assertion follows. \square

Corollary 4.2 *The following statements are equivalent.*

- (1) $E^0({}_R R)$ is flat.
- (2) $E^0(F)$ is flat for any flat left R -module $F.$
- (3) $E^0({}_R R)$ is Gorenstein flat.
- (i)^{op} The opposite version of (i) ($1 \leq i \leq 3$).

Proof. (2) \Leftrightarrow (1) \Leftrightarrow (3) follow from Theorem 4.1 and Lemma 2.5, and (1) \Rightarrow (1)^{op} follows from [Mo, Theorem 1]. Symmetrically, we get the opposite versions of the implications mentioned above. \square

We know from [EJL, Theorem 2.12] that every module in $\text{Mod } R$ has a Gorenstein flat cover. For a module M in $\text{Mod } R$, we denote the Gorenstein flat cover of M by $GF_0(M)$.

Theorem 4.3 *The following statements are equivalent.*

- (1) $E^0({}_R R)$ is Gorenstein flat.
- (2) $E^0(F)$ is Gorenstein flat for any flat left R -module F .
- (3) $E^0(G)$ is Gorenstein flat for any Gorenstein flat left R -module G .
- (4) $GF_0(M)$ is injective for any 1-Gorenstein cotorsion left R -module M .
- (5) $GF_0(M)$ is injective for any strongly Gorenstein cotorsion left R -module M .
- (6) $GF_0(E)$ is injective for any injective left R -module E .
- (7) $E^0(N)$ is flat for any 1-Gorenstein torsionfree right R -module N .
- (8) $E^0(N)$ is Gorenstein flat for any 1-Gorenstein torsionfree right R -module N .
- (9) $E^0(N)$ is flat for any strongly Gorenstein torsionfree right R -module N .
- (10) $E^0(N)$ is Gorenstein flat for any strongly Gorenstein torsionfree right R -module N .

Proof. (1) \Rightarrow (2) By Corollary 4.2.

(2) \Rightarrow (3) Let $G \in \text{Mod } R$ be Gorenstein flat. From the proof of Theorem 4.1, we know that there exists a flat left R -module F such that $E^0(G)$ is isomorphic to a direct summand of $E^0(F)$. By (2), $E^0(F)$ is Gorenstein flat. Because $\mathcal{GF}(R)$ is closed under direct summands by Lemma 2.6, $E^0(G)$ is also Gorenstein flat.

(3) \Rightarrow (4) Let $M \in \text{Mod } R$ be 1-Gorenstein cotorsion. Consider the following push-out diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & X & \longrightarrow & GF_0(M) & \longrightarrow & M \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & X & \longrightarrow & E_0(GF_0(M)) & \longrightarrow & N \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & T & \xlongequal{\quad} & T \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

By (3), $E^0(GF_0(M))$ is Gorenstein flat. Then by the exactness of the middle column in

the above diagram, we have that the Gorenstein flat dimension of T is at most one. So $\text{Ext}_R^1(T, M) = 0$ and the rightmost column $0 \rightarrow M \rightarrow N \rightarrow T \rightarrow 0$ in the above diagram splits, which implies that M is isomorphic to a direct summand of N . It follows that $GF_0(M)$ is isomorphic to a direct summand of $GF_0(N)$.

Note that $\mathcal{GF}(R)$ is closed under extensions by Lemma 2.6. So $\text{Ext}_R^1(G, X) = 0$ for any Gorenstein flat left R -module G by Lemma 2.3, which implies that $E^0(GF_0(M))$ is a Gorenstein flat precover of N by the exactness of the middle row in the above diagram. Thus $GF_0(N)$ is isomorphic to a direct summand of $E^0(GF_0(M))$, and therefore $GF_0(N)$ and $GF_0(M)$ are injective.

(4) \Rightarrow (5) \Rightarrow (6) are trivial.

(6) \Rightarrow (1) Consider the following pull-back diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & W & \xlongequal{\quad} & W & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & GF_0(E^0({}_R R)) & \longrightarrow & H \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & {}_R R & \longrightarrow & E^0({}_R R) & \longrightarrow & H \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

Then ${}_R R$ is isomorphic to a direct summand of Y , and so $E^0({}_R R)$ is isomorphic to a direct summand of $E^0(Y)$. Because $GF_0(E^0({}_R R))$ is injective by (6), the exactness of the middle row in the above diagram implies that $E^0(Y)$ is isomorphic to a direct summand of $GF_0(E^0({}_R R))$. Thus $E^0({}_R R)$ is also isomorphic to a direct summand of $GF_0(E^0({}_R R))$. Again by Lemma 2.6, $\mathcal{GF}(R)$ is closed under direct summands, so we get that $E^0({}_R R)$ is Gorenstein flat.

(1) + (4) \Rightarrow (7) Let $N \in \text{Mod } R^{op}$ be 1-Gorenstein torsionfree. Then $N^+ \in \text{Mod } R$ is 1-Gorenstein cotorsion by Proposition 2.10, and so $GF_0(N^+)$ is injective by (4).

From the epimorphism $GF_0(N^+) \twoheadrightarrow N^+$ in $\text{Mod } R$, we get a monomorphism $N^{++} \hookrightarrow [GF_0(N^+)]^+$ in $\text{Mod } R^{op}$ with $[GF_0(N^+)]^+$ flat by Lemma 2.12(2). Then $E^0(N^{++})$ is isomorphic to a direct summand of $E^0([GF_0(N^+)]^+)$. On the other hand, there exists a monomorphism $N \hookrightarrow N^{++}$ by [S, p.48, Exercise 41], $E^0(N)$ is isomorphic to a direct summand of $E^0(N^{++})$, and hence a direct summand of $E^0([GF_0(N^+)]^+)$. Because $E^0([GF_0(N^+)]^+)$ is

flat by (1) and Corollary 4.2, $E^0(N)$ is also flat.

(7) \Rightarrow (8) \Rightarrow (10) and (7) \Rightarrow (9) \Rightarrow (10) are trivial.

(10) \Rightarrow (1) By (10), we have that $E^0(R_R)$ is Gorenstein flat. Then $E^0({}_R R)$ is also Gorenstein flat by Corollary 4.2. \square

The following result is an analogue of Theorem 4.3.

Theorem 4.4 *The following statements are equivalent.*

- (1) $E^0({}_R R)$ is flat.
- (2) $E^0(F)$ is flat for any flat left R -module F .
- (3) $E^0(G)$ is flat for any Gorenstein flat left R -module G .
- (4) $F_0(M)$ is injective for any 1-cotorsion left R -module M .
- (5) $F_0(M)$ is injective for any strongly cotorsion left R -module M .
- (6) $F_0(E)$ is injective for any injective left R -module E .
- (7) $E^0(N)$ is flat for any 1-torsionfree right R -module N .
- (8) $E^0(N)$ is Gorenstein flat for any 1-torsionfree right R -module N .
- (9) $E^0(N)$ is flat for any strongly torsionfree right R -module N .
- (10) $E^0(N)$ is Gorenstein flat for any strongly torsionfree right R -module N .

Proof. The proof is similar to that of Theorem 4.3, so we omit it. \square

Putting the results in this section and their opposite versions together we have the following

Theorem 4.5 *The following statements are equivalent.*

- (1) $E^0({}_R R)$ is flat.
- (2) $E^0(G)$ is flat for any (Gorenstein) flat left R -module G .
- (3) $E^0(M)$ is (Gorenstein) flat for any 1-torsionfree left R -module M .
- (4) $E^0(M)$ is (Gorenstein) flat for any strongly torsionfree left R -module M .
- (5) $F_0(M)$ is injective for any 1-cotorsion left R -module M .
- (6) $F_0(M)$ is injective for any strongly cotorsion left R -module M .
- (7) $F_0(E)$ is injective for any injective left R -module E .
- (i)^{op} The opposite version of (i) ($1 \leq i \leq 7$).
- (G1) $E^0({}_R R)$ is Gorenstein flat.
- (G2) $E^0(G)$ is Gorenstein flat for any (Gorenstein) flat left R -module G .
- (G3) $E^0(M)$ is (Gorenstein) flat for any 1-Gorenstein torsionfree left R -module M .
- (G4) $E^0(M)$ is (Gorenstein) flat for any strongly Gorenstein torsionfree left R -module M .

- (G5) $GF_0(M)$ is injective for any 1-Gorenstein cotorsion left R -module M .
(G6) $GF_0(M)$ is injective for any strongly Gorenstein cotorsion left R -module M .
(G7) $GF_0(E)$ is injective for any injective left R -module E .
 $(Gi)^{op}$ The opposite version of (Gi) ($1 \leq i \leq 7$).

Consider the following conditions for any $n \geq 0$.

- (1) $\text{fd}_R E^0({}_R R) \leq n$.
(2) $\text{id}_R F_0(E) \leq n$ for any injective left R -module E .

When $n = 0$, (1) \Leftrightarrow (2) by Theorem 4.5. However, when $n \geq 1$, neither “(1) \Rightarrow (2)” nor “(2) \Rightarrow (1)” hold true in general as shown in the following example.

Example 4.6 Let R be a finite-dimensional algebra over a field K and Δ the quiver:

$$1 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 2 \xrightarrow{\gamma} 3.$$

(1) If $R = K\Delta/(\alpha\beta\alpha)$, then $\text{fd}_R E^0({}_R R) = 1$ and $\text{fd}_{R^{op}} E^0(R_R) \geq 2$. We have that $\mathbb{D}[E^0(R_R)] \rightarrow \mathbb{D}(R_R)$ is the flat cover of the injective left R -module $\mathbb{D}(R_R)$ with $\text{id}_R \mathbb{D}[E^0(R_R)] \geq 2$, where $\mathbb{D}(-) = \text{Hom}_K(-, K)$.

(2) If $R = K\Delta/(\gamma\alpha, \beta\alpha)$, then $\text{fd}_R E^0({}_R R) = 2$ and $\text{fd}_{R^{op}} E^0(R_R) = 1$. We have that $\mathbb{D}[E^0(R_R)] \rightarrow \mathbb{D}(R_R)$ is the flat cover of the injective left R -module $\mathbb{D}(R_R)$ with $\text{id}_R \mathbb{D}[E^0(R_R)] = 1$. Because $\mathbb{D}(R_R)$ is an injective cogenerator for $\text{Mod } R$, $\text{id}_R F_0(E) \leq 1$ for any injective left R -module E .

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